Algebraic geometry 2

Exercise Sheet 10

PD Dr. Maksim Zhykhovich Summer Semester 2026, 18.06.2026

Finish all exercises from Sheet 9.

Exercise 1. Let $\varphi : A \to B$ be a ring homomorphism and let
$f : \operatorname{Spec} B \to \operatorname{Spec} A$ be a respective map of schemes.
(1) Show that φ is injective if and only if the respective morphism of sheaves $f^\sharp$ is injective.
(2) Show that: if φ is injective then f is dominant (that is, the image of f is dense in $\operatorname{Spec} A$.

Exercise 2. Consider the affine line $\mathbb{A}^1_{\mathbb{Z}} := \operatorname{Spec} \mathbb{Z}[t]$ over $\mathbb{Z}$.
(1) Let p be a prime number. Show that the ideal $\mathfrak{m} = (p, t) \subset \mathbb{Z}[X]$ defines a closed point in $\mathbb{A}^1_{\mathbb{Z}}$.
(2) Let $U = \mathbb{A}^1_{\mathbb{Z}} \backslash \{\mathfrak{m}\}$. Show that $U = D(p) \cup D(t)$ and that the restriction map $\mathbb{Z}[t] \simeq \mathcal{O}_X(X) \to \mathcal{O}_X(U)$ is an isomorphism, where $X = \mathbb{A}^1_{\mathbb{Z}}$.
(3) Show that U is not an affine scheme.
(4) Consider now the affine line $\mathbb{A}^1_k := \operatorname{Spec} k[t]$ over a field k. Let x be a closed point of $\mathbb{A}^1_k$. Show that $\mathbb{A}^1_k \backslash \{x\}$ is an affine scheme.